$\boxed{1}$ 曲面 $\overrightarrow{r}(u,v)=(u\cos v, u\sin v, v)$ $(0\leq v\leq u\leq 1)$ の面積を求めよ。

まず、

$$D=\{(u,v)\,;\,0\leq u\leq 1\,,\,0\leq v\leq u\}.$$

$$\begin{aligned}\overrightarrow{r}_u&=(\cos v\,,\,\sin v\,,\,0),\\ \overrightarrow{r}_v&=(-u\sin v\,,\,u\cos v\,,\,1).\end{aligned}$$

$$\begin{aligned}\overrightarrow{r}_u\times\overrightarrow{r}_v&=(\sin v-0\,,\,0-\cos v\,,\,u\cos^2 v+u\sin^2 v)\\ &=(\sin v\,,\,-\cos\,,\,u),\\ |\overrightarrow{r}_u\times\overrightarrow{r}_v|&=\sqrt{1+u^2}.\end{aligned}$$

よって求める面積は、

$$\begin{aligned}S&=\iint_D|\overrightarrow{r}_u\times\overrightarrow{r}_v|\,du\,dv\\ &=\int_0^1\left(\int_0^u\sqrt{1+u^2}\,dv\right)\,du\\ &=\int_0^1 u\sqrt{1+u^2}\,du\\ &=\int_0^1\left(\frac{1}{3}(1+u^2)^{\frac{3}{2}}\right)'\,du\\ &=\frac{2\sqrt{2}-1}{3}.\end{aligned}$$

$\boxed{2}$ 曲面 $\overrightarrow{r}(u,v) = (u+v, uv, u-v)$ $(u^2+v^2 \leq 1,\, u \geq 0,\, v \geq 0)$ の面積を求めよ。

まず、

$$u = l\cos\theta\,,\ v = l\sin\theta \qquad \left(0 \leq l \leq 1\,,\ 0 \leq \theta \leq \frac{\pi}{2}\right)$$

とおくと

$$\overrightarrow{r}(l,\theta) = \left(l(\cos\theta+\sin\theta)\,,\ \frac{l^2}{2}\sin 2\theta\,,\ l(\cos\theta-\sin\theta)\right)$$

よって

$$\begin{aligned}\overrightarrow{r}_l &= (\cos\theta+\sin\theta\,,\ l\sin 2\theta\,,\ \cos\theta-\sin\theta),\\ \overrightarrow{r}_\theta &= \big(l(\cos\theta-\sin\theta)\,,\ l^2\cos 2\theta\,,\ -l(\cos\theta+\sin\theta)\big).\end{aligned}$$

$$\begin{aligned}\overrightarrow{r}_l \times \overrightarrow{r}_\theta &= \big(-l^2\sin 2\theta(cos\theta+\sin\theta) - l^2\cos 2\theta(\cos\theta-\sin\theta)\,,\\ &\quad l(\cos\theta-\sin\theta)^2 + l(\cos\theta+\sin\theta)^2\,,\\ &\quad l^2\cos 2\theta(\cos\theta+\sin\ \theta) - l^2\sin 2\theta(\cos\theta-\sin\theta)\big)\\ &= \big(-l^2\{\cos(2\theta-\theta)+\sin(2\theta-\theta)\}\,,\\ &\quad 2l(\cos^2\theta+\sin^2\theta)\,,\\ &\quad l^2\{\cos(2\theta-\theta)-\sin(2\theta-\theta)\}\big)\\ &= \big(-l^2(\cos\theta+\sin\theta)\,,\ 2l\,,\ l^2(\cos\theta-\sin\theta)\big),\\ |\overrightarrow{r}_l \times \overrightarrow{r}_\theta| &= l\sqrt{2l^2+4}.\end{aligned}$$

以上より求める面積は、

$$\begin{aligned}S &= \iint_D |\overrightarrow{r}_l \times \overrightarrow{r}_\theta|\,dl\,d\theta\\ &= \int_0^{\frac{\pi}{2}} d\theta \int_0^1 l\sqrt{2l^2+4}\,dl\\ &= \frac{\pi}{2}\int_0^1 \left(\frac{1}{6}(2l^2+4)^{\frac{3}{2}}\right)' dl\\ &= \frac{3\sqrt{6}-4}{6}\pi.\end{aligned}$$