$\boxed{1}$ 原点 O と 3 点 $A(1,1,1)$, $B(1,2,1)$, $C(2,1,1)$ を頂点とする三角錐 $OABC$ の各面（4 面あり）の外向き面積ベクトルを求め、それらの和が 0 であることを確かめよ。

各面の外向き面積ベクトルはそれぞれ、

$$\begin{aligned} 2\overrightarrow{\triangle ACB} &= \overrightarrow{AC} \times \overrightarrow{AB} = (0,0,1), \\ 2\overrightarrow{\triangle OCA} &= \overrightarrow{OC} \times \overrightarrow{OA} = (0,-1,1), \\ 2\overrightarrow{\triangle OCB} &= \overrightarrow{OB} \times \overrightarrow{OC} = (1,1,-3), \\ 2\overrightarrow{\triangle OBA} &= \overrightarrow{OA} \times \overrightarrow{OB} = (-1,0,1). \end{aligned}$$

よってこれらの和は

$$(0,0,1)+(0,-1,1)+(1,1,-3)+(-1,0,1)=(0,0,0).$$

$\boxed{2}$ 変数変換 $(x,y)=(u\cos v, u\sin v)$ と $(u,v)=(s+t, st)$ の合成変換を求めよ。また、それぞれの微分行列を計算し、

$$\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{pmatrix}$$

が成り立つことを確かめよ。

合成変換は

$$(x,y)=(u\cos v, u\sin v)=((s+t)\cos st, (s+t)\sin st).$$

それぞれの微分行列が

$$\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} \cos st-(s+t)t\sin st & \cos st-(s+t)s\sin st \\ \sin st+(s+t)s\cos st & \sin st+(s+t)s\cos st \end{pmatrix}.$$

$$\begin{aligned} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{pmatrix} &= \begin{pmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{pmatrix} \begin{pmatrix} 1 & 1 \\ t & s \end{pmatrix} \\ &= \begin{pmatrix} \cos v - ut\sin v & \cos v - us\sin v \\ \sin v + ut\cos v & \sin v + us\cos v \end{pmatrix} \\ &= \begin{pmatrix} \cos st-(s+t)t\sin st & \cos st-(s+t)s\sin st \\ \sin st+(s+t)s\cos st & \sin st+(s+t)s\cos st \end{pmatrix}. \end{aligned}$$

よって

$$\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{pmatrix}.$$